QUALITATIVE INVESTIGATION OF A SYSTEM OF DIFFERENTIAL EQUATIONS OF THE THEORY OF OSCILLATIONS

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In this note there are given the results of a qualitative study of the system

$$\frac{dx_1}{dt_1} = x_1(mx_1 + ny_1 + p), \qquad \frac{dy_1}{dt_1} = y_1(a'x_1 + b'y_1 + c') \tag{0.1}$$

which arises in the theory of oscillations. Various particular cases of the system (0.1) have been investigated in a number of works. Applications of the method of Van der Pol to the action of an external force on a system with two degrees of freedom near a linear conservative system [1-5], for example, lead to the system (0.1). Certain problems of chemical kinetics [6-8], of astrophysics [9,10], of mathematical biology [11-13], and of other fields can also be reduced to the solution of the system (0.1).

Jones [10] made some incorrect statements on the behavior of the separatrix, which led the author of that paper to false conclusions on the possibility of the existence of limit cycles for the system (0.1). A proof of the absence of limit cycles for the system (0.1) is given in [14].

1. None of the coefficients m, n and p vanishes. Making use of the transformation

$$x_1 = \frac{p}{m}x, \qquad y_1 = \frac{p}{n}y, \qquad t_1 = \frac{t}{p}$$

we can reduce the system (0.1) to the form

$$\frac{dx}{dt} = x(x+y+1), \qquad \frac{dy}{dt} = y(ax+by+c) \tag{1.1}$$

Eliminating t, we obtain

$$\frac{dy}{dx} = \frac{y(ax+by+c)}{x(x+y+1)} \tag{1.2}$$

The points

$$P_1(0,0), \quad P_2\left(0, -\frac{c}{b}\right), \quad P_3(-1,0), \quad P_4\left(\frac{c-b}{b-a}, \frac{a-c}{b-a}\right) \text{ when } \Delta = \begin{vmatrix} 1 & 1 \\ a & b \end{vmatrix} \neq 0$$

represent the state of equilibrium of the system (1.1).

For the investigation of the nature of these points we find the roots λ_1 and λ_2 of the corresponding characteristic equations

$$\lambda_1 = c, \quad \lambda_2 = 1 \text{ for the point } P_1$$
$$\lambda_1 = -c, \quad \lambda_2 = (b-c)/b \text{ for the point } P_2$$
$$\lambda_1 = -1, \quad \lambda_2 = c-a \text{ for the point } P_3$$
$$\lambda_{1,2} = \frac{ab-bc+c-b \pm \sqrt{(ab-bc+c-b)^2 - 4(b-a)(c-b)(a-c)}}{2(b-a)} \text{ for the point } P_4$$

The integral curves (1.2) pass through the points P_1, P_2 and P_3. These points can therefore be only nodes, or saddles.

If $(b-a)(c-b)(a-c) < 0$, the point P_4 will be a point of equilibrium of the saddle type. If, however, $(b-a)(c-b)(a-c) > 0$, and $ab - bc + c - b = 0$, then the system (1.1) has a center [14] at the point P_4.

For the purpose of revealing the behavior of the trajectories at infinity, we shall map the phase plane onto the sphere of Poincaré. Performing the transformation $x = 1/z$, $y = \tau/z$, we obtain

$$\frac{dz}{d\tau} = \frac{-z(\tau+z+1)}{\tau[(b-1)\tau + (c-1)z + a - 1]}$$

Examining this equation, we find four points P_5, P_5', P_6 and P_6' on the equator of the sphere, which are pair-wise diametrically opposite to each other. The points P_5 and P_5' correspond to the positive and negative "ends", respectively, of the x-axis, while the points P_6 and P_6' are located at the ends of the diameter whose angular coefficient is equal to $(a-1)/(1-b)$ (we assume that the point P_6 lies on the right half-plane).

Finding the roots λ_1 and λ_2 of the corresponding characteristic equations, we obtain

$$\lambda_1 = -1, \ \lambda_2 = a - 1 \quad \text{for } P_5; \qquad \lambda_1 = 1 - a, \ \lambda_2 = (a - b)/(b - 1) \quad \text{for } P_6$$

Performing the transformation $x = \tau/z$, $y = 1/z$, one can easily convince oneself that on the equator there exist still two special (singular) points P_7 and P_7' which coincide with the positive and negative ends, respectively, of the y-axis. The roots of the characteristic equation for the point P_7 are $\lambda_1 = -b$, $\lambda_2 = 1 - b$.

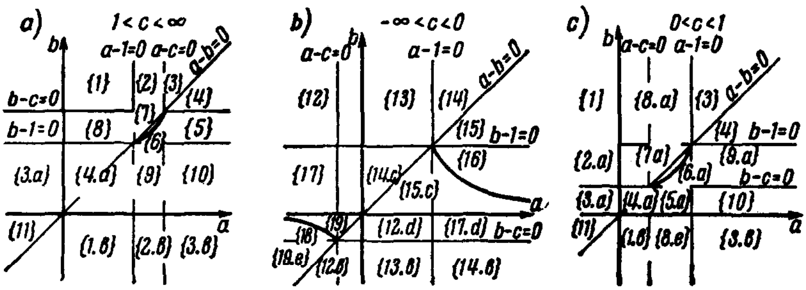

Fig. 1.

We shall consider the dependence of the qualitative picture of the phase trajectories of the system (1.1) on the parameters. Fixing the parameter c, and drawing on the plane of the parameters a and b the lines $a - b = 0$, $a - 1 = 0$, $a - c = 0$, $b - 1 = 0$, $b - c = 0$, and $ab - bc + c - b = 0$ when $(b - a)(c - b)(a - c) > 0$, which correspond to the bifurcation values of the parameters a and b, we obtain a division of the plane a, b into regions, to each of which there corresponds a definite qualitative picture of the breaking up of the trajectory (Fig. 1) of the lower hemisphere of Poincaré for the system (1.1). Hereby it is necessary to consider three cases: (1) $1 < c < \infty$, (2) $-\infty < c < 0$, (3) $0 < c < 1$.

The results of the investigation of the special points $P_1, \ldots, P_7'$ for each of these cases are given in Table 1.*

* In the tables we use the following notation: α_1 is a stable node, α_2 is an unstable node, β_1 is a stable focus, β_2 is an unstable focus, γ is a saddle, $\gamma\alpha$ is a saddle-node, δ is a complicated singular point which is obtained when four coarse singular points merge.

When $0 < c < 1$, the qualitative picture of the phase trajectories in the regions

$$\{1\}\ (-\infty < a < c,\ 1 < b < \infty),\quad \{3\}\ (1 < a < b,\ 1 < b < \infty),\quad \{4\}\ (b < a < \infty, 1 < b < \infty)$$
$$\{3a\}\ (-\infty < a < b,\ 0 < b < c),\quad \{4a\}\ (b < a < c,\ 0 < b < c),\quad \{10\}\ (1 < a < \infty, 0 < b < c)$$
$$\{3b\}\ (1 < a < \infty,\ -\infty < b < 0),\quad \{1, b\}\ (b < a < c,\ -\infty < b < 0),$$
$$\{11\}\ (-\infty < a < b,\ -\infty < b < 0)$$

are the same as in the corresponding regions when $1 < c < \infty$.

Suppose, furthermore, that

$$\Delta = \begin{vmatrix} 1 & 1 \\ a & b \end{vmatrix} = 0 \quad \text{or} \quad a = b$$

Then, if $a = b \neq 1$, the point P_4 will pass onto the equator, and it will form there a complicated singular point of the type of a saddle-node.

Let us consider the case when $a = b = 1$. The points $P_1(0, 0)$, $P_2(0, -c)$, $P_3(-1, 0)$ are the state of equilibrium of the system (1.1). It is easy to see that the equator will not be an integral curve in this case. In Table 2 there are given the results of the investigation of the singular points of the system (1.1) for $a = b = 1$.

TABLE 1.

№	Regions	Points									
		P_1	P_2	P_3	P_4	P_5	P_5'	P_6	P_6'	P_7	P_7'
	$1 < c < \infty$										
{1}	$-\infty < a < 1,\ c < b < \infty$	α_2	γ	γ	$\alpha_1\ (\beta_1)$	α_1	α_2	γ	γ	α_1	α_2
{2}	$1 < a < c,\ c < b < \infty$	α_2	γ	γ	$\alpha_1\ (\beta_1)$	γ	γ	α_1	α_2	α_1	α_2
{3}	$c < a < b,\ c < b < \infty$	α_2	γ	α_1	γ	γ	γ	α_1	α_2	α_1	α_2
{4}	$b < a < \infty,\ c < b < \infty$	α_2	γ	α_1	$\alpha_1\ (\beta_1)$	γ	γ	γ	γ	α_1	α_2
{5}	$c < a < \infty,\ 1 < b < c$	α_2	α_1	α_1	γ	γ	γ	γ	γ	α_1	α_2
{6}	$b < a < c,\ 1 < b < c$	α_2	α_1	γ	* 1, 2, 3	γ	γ	γ	γ	α_1	α_2
{7}	$1 < a < b,\ 1 < b < c$	α_2	α_1	γ	γ	γ	γ	α_1	α_2	α_1	α_2
{8}	$-\infty < a < 1,\ 1 < b < c$	α_2	α_1	γ	γ	α_1	α_2	γ	γ	α_1	α_2
{9}	$1 < a < c,\ 0 < b < 1$	α_2	α_1	γ	$\alpha_1\ (\beta_1)$	γ	γ	α_1	α_2	γ	γ
{10}	$c < a < \infty,\ 0 < b < 1$	α_2	α_1	α_1	γ	γ	γ	α_1	α_2	γ	γ
{11}	$-\infty < a < b,\ -\infty < b < 0$	α_2	γ	γ	γ	α_1	α_2	α_2	α_1	α_2	α_1
{1. b}	$b < a < 1,\ -\infty < b < 0$	α_2	γ	γ	$\alpha_1\ (\beta_1)$	α_1	α_2	γ	γ	α_2	α_1
{2. b}	$1 < a < c,\ -\infty < b < 0$	α_2	γ	γ	$\alpha_1\ (\beta_1)$	γ	γ	α_1	α_2	α_2	α_1
{3. a}	$-\infty < a < b,\ 0 < b < 1$	α_2	α_1	γ	γ	α_1	α_2	α_2	α_1	γ	γ
{3. b}	$c < a < \infty,\ -\infty < b < 0$	α_2	γ	α_1	γ	γ	γ	α_1	α_2	α_2	α_1
{4. a}	$b < a < 1,\ 0 < b < 1$	α_2	α_1	γ	$\alpha_1\ (\beta_1)$	α_1	α_2	γ	γ	γ	γ

Table 1 *contd.:*

№	Regions	Points									
		P_1	P_2	P_3	P_4	P_5	P_5'	P_6	P_6'	P_7	P_7'
	$-\infty < c < 0$										
{12}	$-\infty < a < c,\ 1 < b < \infty$	γ	α_2	γ	$\alpha_1\ (\beta_1)$	α_1	α_2	γ	γ	α_1	α_2
{13}	$c < a < 1,\ 1 < b < \infty$	γ	α_2	α_1	γ	α_1	α_2	γ	γ	α_1	α_2
{14}	$1 < a < b,\ 1 < b < \infty$	γ	α_2	α_1	γ	γ	γ	α_1	α_2	α_1	α_2
{15}	$b < a < \infty,\ 1 < b < \infty$	γ	α_2	α_1	$\alpha_1\ (\beta_1)$	γ	γ	γ	γ	α_1	α_2
{16}	$1 < a < \infty,\ 0 < b < 1$	γ	α_2	α_1	* 1, 2, 3	γ	γ	α_1	α_2	γ	γ
{17}	$-\infty < a < c,\ 0 < b < 1$	γ	α_2	γ	$\alpha_1\ (\beta_1)$	α_1	α_2	α_2	α_1	γ	γ
{18}	$-\infty < a < c,\ c < b < 0$	γ	γ	γ	** 1, 2, 3	α_1	α_2	α_2	α_1	α_2	α_1
{19}	$c < a < b,\ c < b < 0$	γ	γ	α_1	γ	α_1	α_2	α_2	α_1	α_2	α_1
{12. *b*}	$b < a < c,\ -\infty < b < c$	γ	α_2	γ	$\alpha_1\ (\beta_1)$	α_1	α_2	γ	γ	α_2	α_1
{12. *d*}	$b < a < 1,\ c < b < 0$	γ	γ	α_1	$\alpha_2\ (\beta_2)$	α_1	α_2	γ	γ	α_2	α_1
{13. *b*}	$c < a < 1,\ -\infty < b < c$	γ	α_2	α_1	γ	α_1	α_2	γ	γ	α_2	α_1
{14. *b*}	$1 < a < \infty,\ -\infty < b < c$	γ	α_2	α_1	γ	γ	γ	α_1	α_2	α_2	α_1
{14. *c*}	$c < a < b,\ 0 < b < 1$	γ	α_2	α_1	γ	α_1	α_2	α_2	α_1	γ	γ
{15. *c*}	$b < a < 1,\ 0 < b < 1$	γ	α_2	α_1	$\alpha_2\ (\beta_2)$	α_1	α_2	γ	γ	γ	γ
{17. *d*}	$1 < a < \infty,\ c < b < 0$	γ	γ	α_1	$\alpha_2\ (\beta_2)$	γ	γ	α_1	α_2	α_2	α_1
{19. *e*}	$-\infty < a < b,\ -\infty < b < c$	γ	α_2	γ	γ	α_1	α_2	α_2	α_1	α_2	α_1
	$0 < c < 1$										
{2. *a*}	$-\infty < a < c,\ c < b < 1$	α_2	γ	γ	$\alpha_1\ (\beta_1)$	α_1	α_2	α_2	α_1	γ	γ
{5. *a*}	$c < a < 1,\ 0 < b < c$	α_2	α_1	α_1	γ	α_1	α_2	γ	γ	γ	γ
{6. *a*}	$b < a < 1,\ c < b < 1$	α_2	γ	α_1	* 1, 2, 3	α_1	α_2	γ	γ	γ	γ
{7. *a*}	$c < a < b'\ c < b < 1$	α_2	γ	α_1	γ	α_1	α_2	α_2	α_1	γ	γ
{8. *a*}	$c < a < 1,\ 1 < b < \infty$	α_2	γ	α_1	γ	α_1	α_2	γ	γ	α_1	α_2
{8. *e*}	$c < a < 1,\ -\infty < b < 0$	α_2	γ	α_1	γ	α_1	α_2	γ	γ	α_2	α_1
{9. *a*}	$1 < a < \infty,\ c < b < 1$	α_2	γ	α_1	$\alpha_1\ (\beta_1)$	γ	γ	α_1	α_2	γ	γ

		*	**
1)	stable node (focus) when	$b - c - ab + bc < 0$	> 0
2)	center when	$b - c - ab + bc = 0$	$= 0$
3)	unstable node (focus) when	$b - c - ab + bc > 0$	< 0

2. Some of the coefficients of the first and second equation* of the system (0.1) vanish. a) Suppose that $p = a' = 0$. Performing the transformation

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* The case when some of the coefficients m, n and p vanish, but none of the numbers a', b' and c' vanish, can be reduced to the case considered by means of a change of variables.

$$x_1 = \frac{c'}{m}x, \quad y_1 = \frac{c'}{n}y, \quad t_1 = \frac{1}{c'}t \qquad (2.1)$$

and eliminating t we obtain the equation

$$\frac{dy}{dx} = \frac{y(by+1)}{x(x+y)} \qquad (2.2)$$

TABLE 2.

$a = b = 1$

№	Regions	Points		
		P_1	P_2	P_3
{20}	$-\infty < c < 0$	γ	α_2	α_1
{21}	$0 < c < 1$	α_2	γ	α_1
{21.a}	$1 < c < \infty$	α_2	α_1	γ

The equation (2.2) has three singular points $P_1(0, 0)$, $P_2(0, -1/b)$, and $P_3(1/b, -1/b)$ on the xy-plane. On the equator there exist six pair-wise diametrically opposed points: P_4 and P_4' coinciding with the positive and negative ends, respectively, of the x-axis; P_5 and P_5', located at the ends of the diameter whose angular coefficient is equal to $1/(b-1)$; P_6 and P_6', located at the positive and negative ends, respectively, of the y-axis (we assume that the point P_5 is located on the right half-plane). The results of the study of these points are given in Table 3.

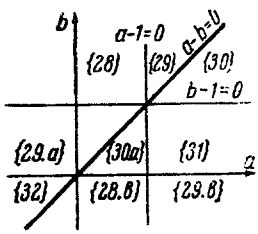

Fig. 2.

b) Suppose that $p = b' = 0$. Performing the transformation (2.1) and eliminating the parameter, we obtain the equation

$$\frac{dy}{dx} = \frac{y(ax+1)}{x(x+y)} \qquad (2.3)$$

On the xy-plane the equation (2.3) has two singular points $P_1(0, 0)$, and $P_2(-1/a, 1/a)$. On the equator there are six pair-wise diametrically opposite points: P_3 and P_3' coinciding with the positive and negative ends, respectively, of the x-axis; P_4 and P_4', located on the ends of the diameters whose angular coefficient is equal to $a - 1$; P_5 and P_5' coinciding with the ends of the positive and negative ends, respectively, of the y-axis (the point P_4 is located on the right half-plane). The results of the study of these points are given in Table 4.

TABLE 3.

№	Regions	Points								
		P_1	P_2	P_3	P_4	P_4'	P_5	P_5'	P_6	P_6'
{22}	$-\infty < b < 0$	$\gamma\alpha$	γ	α_1	α_1	α_2	γ	γ	α_2	α_1
{23}	$0 < b < 1$	$\gamma\alpha$	α_1	γ	α_1	α_2	α_2	α_1	γ	γ
{24}	$1 < b < \infty$	$\gamma\alpha$	α_1	γ	α_1	α_2	γ	γ	α_1	α_2

TABLE 4.

№	Regions	Points							
		P_1	P_2	P_3	P_3'	P_4	P_4'	P_5	P_5'
{25}	$-\infty < a < 0$	$\gamma\alpha$	γ	α_1	α_2	α_2	α_1	γ	α_1
{26}	$0 < a < 1$	$\gamma\alpha$	$\alpha_1\,(\beta_1)$	α_1	α_2	γ	γ	γ	α_1
{27}	$1 < a < \infty$	$\gamma\alpha$	$\alpha_1\,(\beta_1)$	γ	γ	α_1	α_2	γ	α_1

c) Suppose that $p = c' = 0$. Setting $x_1 = x/m$, $y_1 = y/n$ in the system (0.1), and eliminating the parameter, we obtan

$$\frac{dy}{dx} = \frac{y\,(ax + by)}{x\,(x + y)} \tag{2.4}$$

The equation (2.4) has a complicated singular point at the origin of the coordinate system. As in the previous case, there are six points on the equator: P_2 and P_2', the ends of the x-axis; P_3 and P_3' coinciding with the diameter whose angular coefficient is equal to $(a - 1)/(1 - b)$; P_4 and P_4', the ends of the positive and negative parts, respectively, of the y-axis (the point P_2 and P_3 are located on the right half-plane). In Fig. 2 there is represented the plane of the parameters a and b. The results of the study of the singular points in each of the regions of the plane of the parameters a and b are given in Table 5.

3. Results of the investigation. In Figs. 3 and 4 there are given the qualitative pictures of the division of the trajectories of the lower hemisphere of Poincaré for all cases considered. The qualitative pictures of the division for the cases $\{2.a\}$, $\{3.a\}$, $\{4.a\}$, $\{5.a\}$, $\{6.a\}$, $\{7.a\}$, $\{8.a\}$, $\{9.a\}$, $\{21.a\}$, $\{29.a\}$, and $\{30.a\}$ could have been obtained by a rotation through 90° in the clockwise direction and a reflection with respect to the x-axis of the pictures $\{2\}$, $\{3\}$, $\{4\}$, $\{5\}$, $\{6\}$, $\{7\}$, $\{8\}$, $\{9\}$, $\{21\}$, $\{29\}$, and $\{30\}$. The qualitative pictures for the cases $\{1.b\}$, $\{2.b\}$, $\{3.b\}$, $\{12.b\}$, $\{13.b\}$, $\{14.b\}$, $\{28.b\}$, and $\{29.b\}$ can be obtained by means of a reflection with respect to the x-axis of the phase pictures $\{1\}$, $\{2\}$, $\{3\}$ $\{12\}$ $\{13\}$ $\{14\}$ $\{28\}$, and $\{29\}$. The qualitative pictures of the division of the trajectories for the cases $\{14.c\}$ and $\{15.c\}$ can be obtained by means of a rotation through 90° in the counter-clockwise direction and a reflection with respect to the x-axis, from the pictures $\{14\}$ and $\{15\}$ if one hereby also reverses the direction along the trajectory. In an analogous manner one can obtain the pictures for $\{12.d\}$ and $\{17.d\}$ by a rotation through 90° in the counter-clockwise direction of the phase pictures of $\{12\}$ and $\{17\}$ provided one changes the direction along the trajectory. The qualitative picture of the division of the lower hemisphere for the case $\{8.e\}$ can

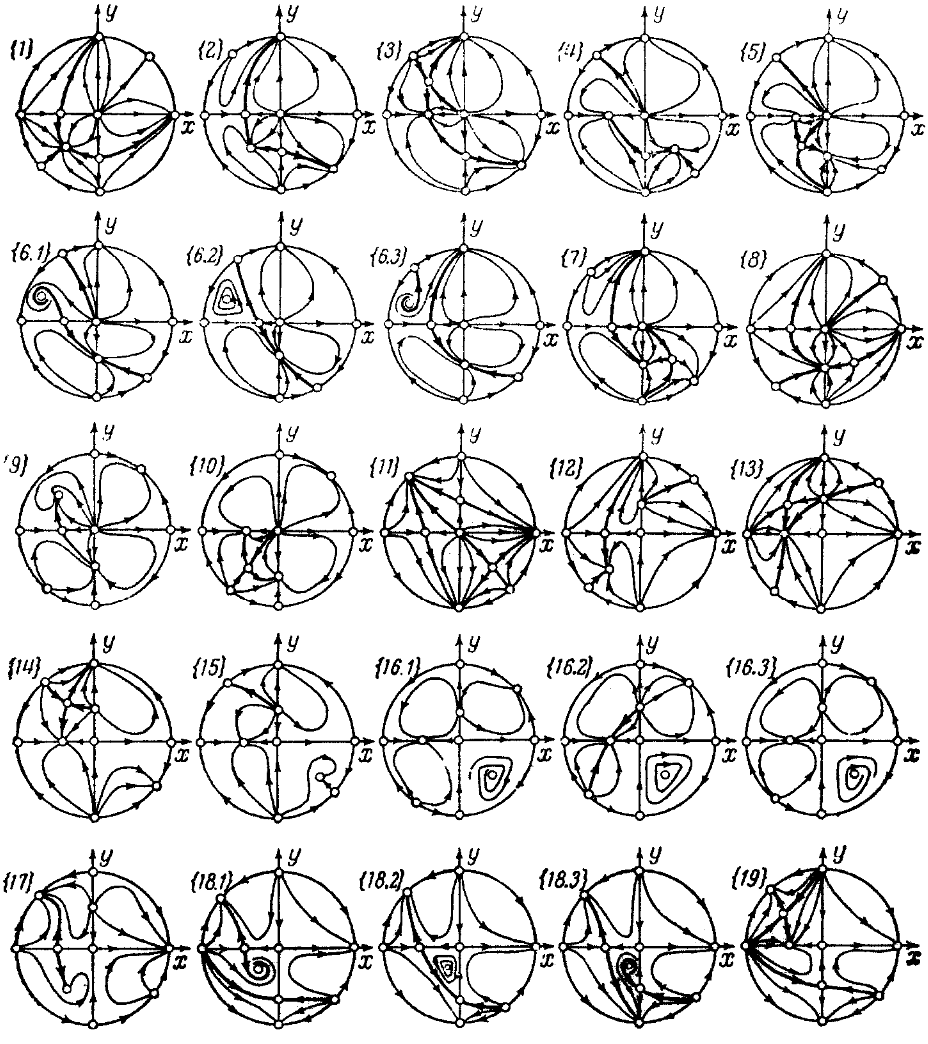

Fig. 3.

be obtained by a rotation through 90° in the clockwise direction of the phase picture $\{8\}$. In order to obtain the qualitative picture of the lower hemisphere for the case $\{19.e\}$, it is necessary to make a reflection with respect to the x-axis of the picture $\{9\}$ and to change the direction along the trajectory, in addition to an original rotation through 90° in the clockwise direction.

TABLE 5.

№	Regions	Points						
		P_1	P_2	P_2'	P_3	P_3'	P_4	P_4'
{28}	$-\infty < a < 1,\ 1 < b < \infty$	δ	α_1	α_2	γ	γ	α_1	α_2
{29}	$1 < a < b,\ 1 < b < \infty$	δ	γ	γ	α_1	α_2	α_1	α_2
{30}	$b < a < \infty,\ 1 < b < \infty$	δ	γ	γ	γ	γ	α_1	α_2
{31}	$1 < a < \infty,\ 0 < b < 1$	δ	γ	γ	α_1	α_2	γ	γ
{32}	$-\infty < a < b,\ -\infty < b < 0$	δ	α_1	α_2	α_2	α_1	α_2	α_1
{28.*b*}	$b < a < 1,\ -\infty < b < 0$	δ	α_1	α_2	γ	γ	α_2	α_1
{29.*a*}	$-\infty < a < b,\ 0 < b < 1$	δ	α_1	α_2	α_2	α_1	γ	γ
{29.*b*}	$1 < a < \infty,\ -\infty < b < 0$	δ	γ	γ	α_1	α_2	α_2	α_1
{30.*a*}	$b < a < 1,\ 0 < b < 1$	δ	α_1	α_2	γ	γ	γ	γ

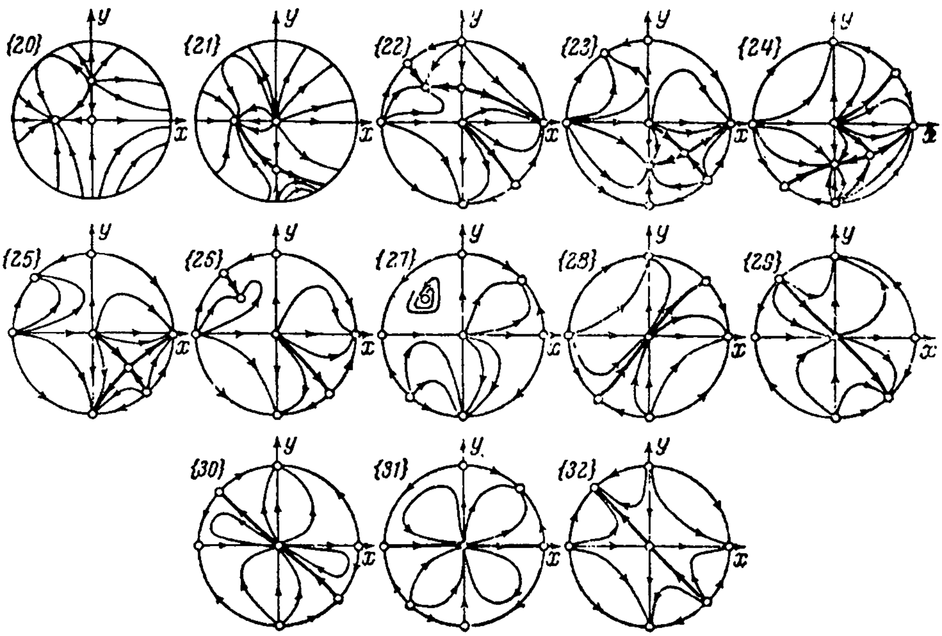

Fig. 4.

In conclusion I thank N.N. Bautin for many suggestions.

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